A primary introduction to spectral theory

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Introduction

This course will be at very elementary level with prerequisite only on limited knowledge of integration theory and linear algebra.

We will start with the theory of Hilbert space and operators on it, focusing our attention on self-adjoint operators and compact operators. After introducing the general notion of spectrum of an operator and its classification, we move on to certain properties of these operators in connection with their spectral behavior, including a simple version of the Fredholm alternative and Weyl's theorem, which is about the invariance of the essential spectrum of an operator under relatively compact perturbation. Finally, we will present an application of the Weyl theorem to an up-to-date research (joint with professor Duyckaerts) concerning the determination of the essential spectrum of a linearized operator, arising as a sum of the Laplace operator and a potential equipped with several properties.

Chapter 1

Hilbert spaces and the spectrum of operators

1.1 Hilbert space and its subspaces

1.1.1 Hilbert spaces

Definition 1.1.1 *A* complex vector space *over the complex field* $\mathbb{C}$ *is a set* $\mathcal{E}$*, whose elements are called* vectors*, with two binary operations, called respectively* addition $(\mathcal{E} \times \mathcal{E} \to \mathcal{E})$ *and* scalar multiplications $(\mathbb{C} \times \mathcal{E} \to \mathcal{E})$*, which are defined as follows:*

To every pair of vectors $x, y \in \mathcal{E}$*, there corresponds a vector* $x + y \in \mathcal{E}$*, in such a manner that* $x + y = y + x$ *and* $x + (y + z) = (x + y) + z$.

$\mathcal{E}$ *contains a unique vector* 0*, called the* zero vector *or* origin *of* $\mathcal{E}$*, such that* $x + 0 = x$ *for every* $x \in \mathcal{E}$*, and to each* $x \in \mathcal{E}$*, there corresponds a unique vector* $-x$ *such that* $x + (-x) = 0$.

To each pair $(\alpha, x) \in \mathbb{C} \times \mathcal{E}$*, there associated a vector* $\alpha x \in \mathcal{E}$*, in such a way that* $1x = x$*,* $\alpha(\beta x) = (\alpha\beta)x$ *and such that the following two distributive laws hold*

$$\alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x.$$

Example 1 *(1).* $\mathcal{E} = \mathbb{R}^n$ *with the usual addition and multiplcation by scalars from* $\mathbb{R}$.

(2). $\mathcal{E} = C([0, 1])$*, all continuous, complex-valued functions on the interval* $[0, 1]$*. Addition is pointwise, that is if* $f, g \in \mathcal{E}$*, then* $(f + g)(x) = f(x) + g(x)$ *and scalar multiplication for* $\mathbb{C}$ *is also pointwise.*

(3). $\mathcal{E} = \ell^2$, *where* ℓ^2 *consists of all infinite sequences of complex numbers:* $x = (x_1, x_2, \ldots, x_n, \ldots)$, *with* $x_n \in \mathbb{C}$ *such that* $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. *Addition and scalar multiplication are defined compenentwise.*

Definition 1.1.2 *A complex vector space* $\mathcal{E}$ *is called an* inner product space *if there is a complex-valued function* $\langle \cdot, \cdot \rangle$ *on* $\mathcal{E} \times \mathcal{E}$ *that satisfies the following three conditions for all* $x, y, z \in \mathcal{E}$ *and* $\alpha \in \mathbb{C}$*:*

(1). $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$ *(positive definiteness)*

(2). $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$ *(linear in the first entry)*

(3). $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *(complex conjugate symmetric)*

If we exclude in the first item of the above definition that $\langle x, x \rangle = 0 \Longleftrightarrow x = 0$, then $\mathcal{E}$ is called *semi-inner product space.*

Example 2 *(1).* $\mathcal{E} = \mathbb{R}^n$ *with* $\langle x, y \rangle := \sum_{j=1}^{n} x_j y_j$ *for* $x, y \in \mathbb{R}^n$.

(2). $\mathcal{E} = C([0,1])$ *with* $\langle f, g \rangle := \int_0^1 f(s)\overline{g(s)} ds$ *for* $f, g \in \mathcal{E}$.

(3). $\mathcal{E} = \ell^2$, *with* $\langle x, y \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$ *for* $x, y \in \ell^2$.

Two vectors x and y, in an inner product space $\mathcal{E}$ is said to be *orthogonal* if $\langle x, y \rangle = 0$. A collection $\{x_j\}$ of vectors of $\mathcal{E}$ is called an *orthonomal set* if $\langle x_j, x_j \rangle = 1$ and $\langle x_j, x_k \rangle = 0$ for all $j \neq k$. We introduce $\|x\| = \sqrt{(x, x)}$.

It is because of the analogy with Euclidean geometry that the geometry of spaces carrying an inner product would be worth-while to investigate. Another reason for us, is the "self-adjointness" to be introduced later.

Proposition 1.1.3 *Let* $\mathcal{E}$ *be an inner product space. Then we have for all* $x, y \in \mathcal{E}$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \tag{1.1}$$

and

$$|(x, y)| \leq \|x\| \cdot \|y\|. \tag{1.2}$$

The equality in (1.2) *holds if and only if there are* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha|^2 + |\beta|^2 \neq 0$ *such that* $\alpha x + \beta y = 0$.

Proof

From the definition of the inner product, (1.1) is clear. For (1.2), if either $x = 0$ or $y = 0$, then both of the two sides equal to zero and there is nothing to prove. Assume $(x, y) \neq 0$. Let $a = \|x\|^2, b = |(x, y)|$, and $c = \|y\|^2$. We may write $(y, x) = b\, e^{i\theta}$ with $i = \sqrt{-1}$, $\theta = \arccos(b^{-1}(y, x))$. For any real number r, we have

$$ar^2 - 2br + c = \|rx - e^{i\theta}y\|^2 \geq 0,$$

which implies $b^2 \leq ac$ and hence (1.2). The equality in (1.2) holds if and only if $ar^2 - 2br + c = 0$ has a unique real root given by $r_0 := b/a$. The proof is complere. ■

We call (1.1) the *parallelogram law* and (1.2) the *Schwarz inequality*.

Proposition 1.1.4 *Let $\mathcal{E}$ be an inner product space. Then we have*

(1). $\|x\| \geq 0$ *and if* $\|x\| = 0$, *then* $x = 0$

(2). $\|\alpha x\| = |\alpha| \cdot \|x\|$, *for all* $\alpha \in \mathbb{C}$

(3). $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathcal{E}$ *(the triangle inequality).*

Proof

The first two properties follow immediately from the definition of the inner product. To prove the triangle inequality, we have by the Schwarz inequality

$$\begin{aligned}\|x + y\|^2 &= (x + y, x + y) = \|x\|^2 + (x, y) + (y, x) + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.\end{aligned}$$

The proof is complere; ■

With the properties (1)-(3) in Proposition 1.1.4, $\|\cdot\|$ is called a *norm*. This norm induces a natural metric on $\mathcal{E}$: $\mathsf{d}(x, y) = \|x - y\|, \quad \forall\, x, y \in \mathcal{E}$.

Definition 1.1.5 *A sequence $\{x_n\}_{n=1}^{\infty}$ of an inner product space $\mathcal{E}$ is said to* converge *to an element $x \in \mathcal{E}$ if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.*

Definition 1.1.6 *A sequence of elements $\{x_n\}_{n=1}^{\infty}$ of an inner product space $\mathcal{E}$ is called a* Cauchy sequence *if for any $\varepsilon > 0$, there exists an $N > 0$ such that $\|x_n - x_m\| < \varepsilon$ holds for all $m, n \geq N$.*

Exercise 1 *Prove that every convergent sequence is a Cauchy sequence. Find an example to show that the converse of this statement is NOT true.*

[Hint: Let $\mathcal{E}$ be the continuous functions on the unit interval $C([0,1])$ equipped with the inner product $(f,g) = \int_0^1 f(x)\overline{g(x)}dx$.]

Definition 1.1.7 *An inner product space $\mathcal{E}$ in which all Cauchy sequences converge is called complete. In this case, $\mathcal{E}$ is called Hilbert space. An inner product space is also called a pre-Hilbert space.*

Remark 1.1.8 *A complete normed space is called Banach space. A Hilbert space, may be of finite, countable, or non-countable dimension. For a given set E, it could be complete with respect to a metric but incomplete with respect to a different metric.*

Example 3 *$\mathcal{E} = L^2(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$, consists of all functions f defined on Ω such that*

$$\|f\|_2 := \left(\int_\Omega |f(x)|^2 dx \right)^{1/2} < \infty.$$

The inner product is defined as $\langle f,g\rangle := \int_\Omega f(x)\overline{g(x)}dx$. One can verify this is a Hilbert space.

Let $\Omega \subset \mathbb{R}^n$ and $C_0^\infty(\Omega)$ be smooth functions vanishing outside a compact subset of Ω. Let $g \in L^2(\Omega)$. We say g has a *weak* or *distributional derivative* with respect to x_j in L^2 if there exists $\phi \in L^2(\Omega)$ such that for any $f \in C_0^\infty(\Omega)$,

$$\int_\Omega f(x)\phi(x)dx = -\int_\Omega \frac{\partial f}{\partial x_j}(x)g(x)dx.$$

For $s \in \mathbb{N}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_j \in \mathbb{N}$, we define $|\sigma| = \sum_{j=1}^n \sigma_j$ and

$$\|f\|_{s,2} := \left(\sum_{|\sigma|\le s} \sum_{j=1}^n \|\partial_{x_j}^{\sigma_j} f\|_2^2 \right)^{1/2}$$

for $f \in C_0^\infty(\Omega)$.

Definition 1.1.9 *The Sobolev space of order s, $H^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_{s,2}$.*

Exercise 2 *Show that $H^s(\Omega)$ is a Hilbert space*

1.1.2 Subspaces

Recall some notions from topology. Let $\mathcal{H}$ be a Hilbert space. The set $\{x \in \mathcal{H} \mid \mathsf{d}(x,y) < r\}$ is called the *open ball*, denoted as $B(y,r)$, of radius r about the point y. A set $O \subset \mathcal{H}$ is called *open* if for any $y \in O$, there exists $r > 0$ such that $B(y,r) \subset O$. A set $N \subset \mathcal{H}$ is called a *neighborhood* of $y \in N$ if there exists $r > 0$ such that $B(y,r) \subset N$. If $G \subset \mathcal{H}$, $x \in G$ is called an *interior point* of G if G is a neighborhood of x. Let $E \subset \mathcal{H}$. A point x is called a *limit point* of E if for any $r > 0$, $B(x,r) \cap (E \setminus \{x\}) \neq \emptyset$. A set $F \subset \mathcal{H}$ is called *closed* if F contains all its limit points. The *closure* $\overline{E}$ of a set $E \subset \mathcal{H}$ is the smallest closed set in $\mathcal{H}$ which contains E. A subset B of $\mathcal{H}$ is called *dense* if every $x \in \mathcal{H}$ is a limit point of B.

A subset W of a Hilbert space $\mathcal{H}$ is called a *subspace* of $\mathcal{H}$ if W is itself a vector space relative to the addition and scaler multiplication which are defined on $\mathcal{H}$. A *closed subspace* of $\mathcal{H}$ is a subspace that is closed set relative to the metric d in $\mathcal{H}$.

Remark 1.1.10 *If $\mathcal{H}$ is of finite dimension, hence an Euclidean space, then every subspace is closed. However, this is not the case when $\mathcal{H}$ is of infinite dimension. Take $\mathcal{H} = L^2([0,1])$ with the natural inner product for example. $C([0,1])$ is a subspace of $\mathcal{H}$, but it is not closed.*

We are more interested in closed subspaces of $\mathcal{H}$ in view of the following fact, which allows us to take the closure of a subspace when necessary.

Proposition 1.1.11 *A closed subspace of a Hilbert space $\mathcal{H}$ is again a Hilbert space. If W is a subspace of $\mathcal{H}$, then its closure $\overline{W}$ is also a subspace of $\mathcal{H}$.*

Exercise 3 *Prove this proposition.*

Definition 1.1.12 *Let W be a closed subspace of a Hilbert space $\mathcal{H}$. The* orthogonal complement *of W in $\mathcal{H}$, denoted as $W^\perp$, is the set of vectors in $\mathcal{H}$ which are orthogonal to W:*

$$W^\perp = \{x \in \mathcal{H} : (x,w) = 0, \forall w \in W\}.$$

We will see later the space $W^\perp$ is always closed.

Theorem 1.1.13 (The projection theorem) *Let W be a closed subspace of $\mathcal{H}$, and let $W^\perp$ be its orthogonal complement. Then $W^\perp$ is a closed subspace of $\mathcal{H}$ and any $x \in \mathcal{H}$ can be uniquely written as $x = y + z$ with $y \in W$ and $z \in W^\perp$.*

Proof

We leave the closedness of $W^\perp$ as an exercise and only show the second part of the theorem.

Step 1. Existence of projection. In this step, we prove that for any $x \in \mathcal{H}$, there exists a unique $y \in W$, called the projection of x on W, such that if $\operatorname{dist}(x, W) = \inf_{w\in W} \|x - w\|$ denotes the distance from x to W, then $\operatorname{dist}(x, W) = \|x - y\|$.

We first show the uniqueness of y given the existence part. Assume there are $y_1, y_2 \in W$ which satisfy

$$\|x - y_1\| = \|x - y_2\| = \operatorname{dist}(x, W) := \delta.$$

Then by (1.1), we have

$$\|y_1 - y_2\|^2 = 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 - 4\left\|\frac{y_1 + y_2}{2} - x\right\|^2,$$

where $\dfrac{y_1 + y_2}{2} \in W$, hence $\left\|\frac{y_1+y_2}{2} - x\right\| \geq \delta$. Therefore, we obtain $\|y_1 - y_2\| \leq 0$.

Next, for the existence part, we note that if $\delta = 0$, then we may take $y = x$. Assume now $\delta > 0$. By definition, there exists $\{w_n\} \subset W$ such that for any $\varepsilon > 0$, there is $N > 0$ such that $\delta \leq \|x - w_n\| < \delta + \varepsilon_1$ for all $n \geq N$ with

$$0 < \varepsilon_1 < \min\Big(\delta, \varepsilon^2/(12\delta)\Big).$$

By the parallelogram law (1.1) again, we have for all $m, n \geq N$

$$\|w_n - w_m\|^2 \leq 4(\delta + \varepsilon_1)^2 - 4\delta^4 = 8\delta\varepsilon_1 + 4\varepsilon_1^2 < \varepsilon^2.$$

Thus $\{w_n\}$ is a Cauchy sequence. Since W is closed, we have w_n converges to some point y in W.

Step 2. End of the proof. Given $x \in \mathcal{H}$, let y be the projection of x on W obtained in Step 1 and put $z = x - y$. We show $z \in W^\perp$ by a perturbation argument as in [5]. Assume there exists a $w \in W \setminus \{0\}$ for which $(z, w) \neq 0$. Consider for any $t \in \mathbb{R}$

$$\delta^2 \leq \|x - (y + tw)\|^2 = \|x - y\|^2 + t^2\|w\|^2 - 2t\,\mathbf{Re}(z, w),$$

or equivalently $0 \leq \big(t\|w\|^2 - 2\,\mathbf{Re}(z, w)\big)t$ holds for all $t \in \mathbb{R}$ and in particular for $t = \|w\|^{-2}\mathbf{Re}(z, w)$. However, this means $0 \leq -\|w\|^{-2}(\mathbf{Re}(z, w))^2$ which implies $(z, w) = 0$ hence a contradiction. Thus $z \in W^\perp$.

To see the uniqueness of z. Assume there are two different ways of writing $x = y_1 + z_1 = y_2 + z_2$ with $y_1, y_2 \in W$ and $z_1, z_2 \in W^\perp$. Then $0 = (y_1 - y_2) + (z_1 - z_2)$ and we have by orthogonality

$$0 = \|(y_1 - y_2) + (z_1 - z_2)\|^2 = \|y_1 - y_2\|^2 + \|z_1 - z_2\|^2.$$

This implies $y_1 = y_2$ and $z_1 = z_2$. The proof is complete. ■

A projection operator or a projector is an operator P for which $P \circ P(x) = P(x)$ holds for all $x \in \mathcal{H}$.

Corollary 1.1.14 *Let W be a subspace of $\mathcal{H}$. Then $W^\perp = \{0\}$ if and only if W is dense in $\mathcal{H}$.*

Using the fact $W^\perp = \overline{W}^\perp$ along with the projection theorem to show the "⇐" part. For the "⇒" part, if $\overline{W} \neq \mathcal{H}$, then we may take $z \in \mathcal{H} \setminus \overline{W}$. Notice that $\overline{W}$ is a closed subspace of $\mathcal{H}$. We may apply the projection theorem to z writing $z = w + h$ in a unique way with $w \in \overline{W}$ and $h \in \overline{W}^\perp = W^\perp$. Since z does not belong to $\overline{W}$, we have $h \neq 0$, a contradiction.

1.2 Operators on Hilbert spaces

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over $\mathbb{C}$. The symbol $T : \mathcal{H}_1 \to \mathcal{H}_2$ is employed to mean that T is a mapping from $\mathcal{H}_1$ to $\mathcal{H}_2$. If $U \subset \mathcal{H}_1$ and $V \subset \mathcal{H}_2$, the *image* of $T(U)$ of U and the *inverse image* or *preimage* $T^{-1}(V)$ of V are defined by

$$T(U) = \{T(x) \mid x \in U\}, \quad T^{-1}(V) = \{x \mid T(x) \in V\}.$$

The mapping T is said to be <u>*linear*</u> if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

holds for all $x, y \in \mathcal{H}_1$ and all scalars $\alpha, \beta \in \mathbb{C}$.

Definition 1.2.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A* linear operator *T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear mapping defined on a linear subspace $D(T) \subset \mathcal{H}_1$, which is called the domain of T, into $\mathcal{H}_2$. When $\mathcal{H}_1$ coincides with $\mathcal{H}_2$, say $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, T is also called a linear operator on $\mathcal{H}$.*

Remark 1.2.2 *In this course, we shall only deal with linear operators. So we adopt a convention that when we say an operator, it is assumed implicitly to be linear.*

Given two linear operators T and S, we define $D(T+S) = D(T) \cap D(S)$ and the *addition* $T+S$ of T ans S on $D(T+S)$ by $(T+S)x = Tx+Sx$. We define the *product* of T and S denoted as $T \circ S$ on the domain $D(T \circ S) = \{x \in D(S) \mid Sx \in D(T)\}$.

Besides the domain of T, there are two more linear subspaces associated to T which play an important role in practice:

$$\operatorname{Ran} T := \{z \in \mathcal{H}_2 \mid z = Tx, \ \text{ for some } x \in D(T)\}$$

which is called the *range* of T, and

$$\text{Ker}\, T := \{x \in D(T) \mid Tx = 0\};$$

which is called the *kernel* of T.

An operator $T^{-1} : \text{Ran}(T) \to D(T)$ is called the *inverse* of T if $T^{-1} \circ T$ is the identity map on $D(T)$ and $T \circ T^{-1}$ is the identity map on $\text{Ran}\, T$.

Remark 1.2.3 *By the Hellinger-Toeplitz theorem, a general unbounded operator T will only be defined on a dense linear subset of $\mathcal{H}$. Thus it is important to specify its domain whenever one talks on an unbounded operator.*

Exercise 4 *Prove that an operator has an inverse if and only if* $\text{Ker}\, T = \{0\}$.

Definition 1.2.4 *A linear operator $T : D(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is called* bounded *if $D(T) = \mathcal{H}_1$ and there exists a $K > 0$ such that for any $x \in \mathcal{H}_1$, we have*

$$\|Tx\|_{\mathcal{H}_2} \leq K\|x\|_{\mathcal{H}_1}.$$

The smallest such K is called the norm *of T, denoted as $\|T\|$, which is given by*

$$\|T\| = \sup_{\|x\|_{\mathcal{H}_1}=1} \|Tx\|_{\mathcal{H}_2}.$$

In the above definition for bounded linear operators, we impose the condition that $D(T) = \mathcal{H}_1$ because if T is continuous with respect to the topology $\|\cdot\|_{\mathcal{H}_2}$, then T has a continuous extension to the closure of $D(T)$ and hence to $\mathcal{H}_1$. This fact manifests in the following theorem (see [5] for a proof.)

Theorem 1.2.5 *Let T be a continous linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then T is bounded and there exists a unique bounded linear operator $\tilde{T}$ with $D(\tilde{T}) = \mathcal{H}_1$ such that $\tilde{T}|_{D(T)} = T$.*

Therefore, it does not make much sense to distinguish T and $\tilde{T}$ when T is bounded.

Definition 1.2.6 *A linear operator T is said to be* invertible *if T has a bounded inverse.*

Definition 1.2.7 *A linear operator $T : D(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is called* densely defined *if $D(T)$ is dense in $\mathcal{H}_1$.*

Example 4 *Let* $\Omega \subset \mathbb{R}^n$ *and consider the Laplacian operator* $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j}$ *acting on* $L^2(\Omega)$*. Then* $D(\Delta) = H^2(\Omega)$*. Notice that* $C_0^\infty(\Omega) \subset H^2(\Omega)$*, so* Δ *is densely defined on* L^2*.*

Definition 1.2.8 *A linear operator* $T : D(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ *is called* continuous *if for each* $\{x_n\} \subset D(T)$ *with* $\lim_{n\to\infty} x_n = x \in D(T)$*, we have* $\lim_{n\to\infty} \|Tx_n - Tx\|_{\mathcal{H}_2} = 0$*.*

Remark 1.2.9 *In this course, we shall only consider densely defined operators. In practice, most useful linear operators are densely defined. Thus in the following context, if* T *is an operator on a Hilbert space* $\mathcal{H}$ *with domain* $D(T)$*, we shall always assume that* $D(T)$ *is dense in* $\mathcal{H}$*.*

An *extension* of an operator T with domain $D(T)$ is an operator $\tilde{T}$ with domain $D(\tilde{T})$, which contains $D(T)$ and the restriction of $\tilde{T}$ to $D(T)$ is equal to T. We use $\tilde{T}|_{D(T)}$ to denote the *restriction* of $\tilde{T}$ to $D(T)$.

Definition 1.2.10 *Let* T *be an operator on* $\mathcal{H}$ *with domain* $D(T)$*. The* adjoint *of* T*, denoted by* T^* *is a map defined on the domain*

$$D(T^*) = \{x \in \mathcal{H} : |\langle Ty, x\rangle| \le C_x \|y\|, 0 < C_x < \infty, \forall y \in D(T)\}$$

such that $T^* : D(T^*) \to \mathcal{H}$ *and*

$$\langle Ty, x\rangle = \langle y, T^* x\rangle \tag{1.3}$$

holds for all $y \in D(T)$ *and* $x \in D(T^*)$*.*

With the assumption that $D(T)$ is dense, it is obvious that T^* is uniquely determined by the relation (1.3). There exist unbounded operators T such that $D(T)$ is dense but $D(T^*)$ is not.

Proposition 1.2.11 (Decomposition of $\mathcal{H}$ via operators) *For any densely defined linear operator* T*, we have*

$$\overline{\text{Ran } T} \oplus \text{Ker } T^* = \mathcal{H}.$$

Proof

First of all, notice that $\overline{\text{Ran } T}$ is a closed subspace of $\mathcal{H}$, we have by the projection Theorem 1.1.13 the unique decomposition $\mathcal{H} = \overline{\text{Ran } T} \oplus (\overline{\text{Ran } T})^\perp$. Now, it suffices to prove $(\text{Ran } T)^\perp = \text{Ker}\, T^*$.

From $(Tx, y) = (x, T^*y) = 0$ for all $x \in D(T)$ and $y \in \text{Ker } T^*$, we have $\text{Ker } T^* \subset (\text{Ran } T)^\perp$. Assume there exists $u \in (\text{Ran } T)^\perp \setminus \text{Ker } T^*$. We have $(Tx, u) = 0$ for all $x \in D(T)$, and in particular $u \in D(T^*)$ by defintion and $(x, T^*u) = 0$ for all $x \in D(T)$. By the assumption that $D(T)$ is dense in $\mathcal{H}$, we have $T^*u = 0$ in view of Corollary 1.1.14. ■

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H} \times \mathcal{H} := \{(x, y) | x, y \in \mathcal{H}\}$ with an inner product defined as follows

$$\langle (x, y), (u, v) \rangle := \langle x, u \rangle + \langle y, v \rangle$$

is a Hilbert space, with topology

$$\|(x, y)\|_{\mathcal{H} \times \mathcal{H}} = \left(\|x\|^2 + \|y\|^2 \right)^{1/2}.$$

Definition 1.2.12 *Let T be a linear operator on $\mathcal{H}$ with domain $D(T)$. The* graph *of T, denoted as $\mathfrak{G}(T)$ is the subset of $\mathcal{H} \times \mathcal{H}$ given by*

$$\mathfrak{G}(T) = \{(x, Tx) | x \in D(T)\}.$$

We say T is closed *if $\mathfrak{G}(T)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. An operator $\bar{T}$ with domain $D(\bar{T})$ is called the* closure *of T if $\mathfrak{G}(\bar{T}) = \overline{\mathfrak{G}(T)}$. An operator T is said to be* closable *if it has a closure. If T is closed, a subset $D \subset D(T)$ is called a* core *for T, if $\overline{T \restriction_D} = T$*

Theorem 1.2.13 *Let T be a linear operator on $\mathcal{H}$. Then T^* is a closed operator.*

Proof

Let $J(x, y) = (-y, x)$ for all $(x, y) \in \mathcal{H} \times \mathcal{H}$. Then we have

$$(u, v) \in \mathfrak{G}(T^*) \Leftrightarrow (Tx, u) = (x, v), \forall x \in D(T) \Leftrightarrow \langle (-Tx, x), (u, v) \rangle = 0, \forall x \in D(T).$$

Thus $\mathfrak{G}(T^*) = (J\mathfrak{G}(T))^\perp$ in $\mathcal{H} \times \mathcal{H}$. ■

A natural question one may ask is whether T^* is densely defined if $D(T)$ is dense. This is not the case if T is an unbounded operator. However, we have the following crucial fact concerning this issue.

Proposition 1.2.14 *$D(T^*)$ is dense if and only if T is closable.*

We skip the proof of this proposition and refer to [5] and [6] for more information about it. This proposition illuminates an important class of operators called self-adjoint operators.

Definition 1.2.15 *An operator T with domain $D(T)$ is called* symmetric *if T^* is an extension of T, or equivalently $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in D(T)$. An operator T is called* self-adjoint *if $T = T^*$, that is T is symmetric and $D(T) = D(T^*)$. A symmetric operator T is called* essentially self-adjoint *if its closure $\overline{T}$ is self-adjoint.*

Clearly from Theorem 1.2.13, a densely defined self-adjoint operator is closed. We will return to the self-adjointness with more properties in the next chapter.

Exercise 5 *Show that for a closable operator T, the closure of T is the unique smallest closed extension of T.*

1.3 Spectrum

Definition 1.3.1 (spectrum) *Let T be a linear operator on $\mathcal{H}$ with domain $D(T)$. The* spectrum *of T, denoted as $\sigma(T)$, is the set of all points $z \in \mathbb{C}$ for which $T - z$ is not invertible*[1]*. If $z \in \sigma(T)$ is such that $\mathrm{Ker}(T - z) \neq \{0\}$, then z is called an* eigenvalue *of T. In this case, any $x \in \mathrm{Ker}(T - z) \setminus \{0\}$ is called an* eigenvector *of $(T; z)$. Moreover, $\dim \mathrm{Ker}(T - z)$ is called the* geometric multiplicity *of z and $\mathrm{Ker}(T - z)$ is called the* geometric eigenspace *of $(T; z)$.*

The discrete spectrum *of T, denoted as $\sigma_d(T)$, is the set of all eigenvalues of T which are isolated points of $\sigma(T)$ and of finite algebraic multiplicity*[2]*.*

The essential spectrum *of T is defined as the complement of $\sigma_d(T)$ in $\sigma(T)$: $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$.*

The essential spectrum is divided into two possible parts called the *continuous spectrum* $\sigma_c(T)$ and the *residual spectrum* $\sigma_r(T)$. $\sigma_c(T)$ consists of all $z \in \mathbb{C}$ for which $T - z$ is a one-to-one mapping from $\mathcal{H}$ onto a dense proper subspace of $\mathcal{H}$. $\sigma_r(T)$ consists of all $z \in \mathbb{C}$ which is not an eigenvalue but $\mathrm{Ran}(T - z)$ is not dense in $\mathcal{H}$. For a wide class of operators, $\sigma_r(T)$ is empty. For example, the self-adjoint operators.

Definition 1.3.2 (resolvent) *Let T be a linear operator on $\mathcal{H}$ with domain $D(T)$. The* resolvent set *of T, denoted as $\rho(T)$, is the set of all points $z \in \mathbb{C}$ for which*

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[1] Be careful that the invertibility includes two ingredients: the existence of an inverse operator along with its boundedness on $\mathcal{H}$.

[2] Recall briefly the algebraic multiplicity. Let z be an eigenvalue of T and consider the sequence $\mathrm{Ker}((T - z)^k)$ for $k = 1, 2, \ldots$ It is proved that there exists a finite p such that $\mathrm{Ker}((T - z)^k)$ stays constant for $k \geq p$, see [1]. The dimension of $\mathrm{Ker}((T - z)^p)$ with the smallest of such a p is called the algebraic multiplicity of z. It coincides with the geometric multiplicity of z if T is self-adjoint.

$T - z$ is invertible. If $z \in \rho(T)$, the inverse of $T - z$ is called the resolvent *of T at z and is written as $R_T(z) = (T - z)^{-1}$.*

Proposition 1.3.3 *Let T be an operator on $\mathcal{H}$. Then for $z, w \in \rho(T)$, we have*

(1). $R_T(z) \circ R_T(w) = R_T(w) \circ R_T(z)$

(2). $R_T(z) - R_T(w) = (z - w) R_T(z) \circ R_T(w)$ (the first resolvent identity)

Proof

Clearly (1) follows from (2). To prove (2), we write

$$R_T(z) - R_T(w) = R_T(z) \circ (T - w) \circ R_T(w) - R_T(z) \circ (T - z) \circ R_T(w),$$

which yields (2). ■

Exercise 6 *Let T, S be two closed operators with $z \in \rho(T) \cap \rho(S)$. Prove the* second resolvent identity

$$R_T(z) - R_S(z) = R_T(z) \circ (S - T) \circ R_S(z) = R_S(z) \circ (S - T) \circ R_T(z).$$

Chapter 2

Self-adjointness

In this chapter, we study serveral properties for self-adjoint operators, which play an important role in the mathematical theory of quantum mechanics. "...It is only for self-adjoint operators that the *spectrum theorem* holds and it is only self-adjoint operators that may be exponentiated to give the one-parameter *unitary groups* which give the dynamics in quantum mechanics"[1]. We will not study the spectrum theorm nor the unitary groups in this course.

2.1 Spectrum of self-adjoint operators

Theorem 2.1.1 *Let T be self-adjoint. Then $\sigma(T) \subset \mathbb{R}$, $\sigma_r(T) = \emptyset$ and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof

Assume $Tf = zf$ with $z = a + ib, b \neq 0$ where a, b are real numbers. Then we have $T - a = T^* - a$ by the self-adjointness of T and hence

$$\|(T - z)f\|^2 = \|(T - a)f\|^2 + b^2\|f\|^2 \geq b^2\|f\|^2. \tag{2.1}$$

First of all, (2.1) implies $\mathrm{Ker}(T-z) = \{0\}$. By self-adjointness of T and Theorem 1.2.11 and Corollary 1.1.14, we have $\mathrm{Ran}(T-z)$ is dense. By the closedness of T and (2.1), one easily obtains the closedness of $\mathrm{Ran}(T - z)$. Therefore, $\mathrm{Ran}(T - z) = \mathcal{H}$ and $T - z$ has an inverse $(T - z)^{-1}$ on $\mathcal{H}$. To show that $(T - z)^{-1}$ is bounded, we may use (2.1) again.

Next we show $\sigma_r(T) = \emptyset$. Since $\sigma(T)$ are real numbers, by the self-adjointness of T, we have $T - \lambda = T^* - \lambda$ for all $\lambda \in \sigma_r(T)$. Moreover, by definition of the

[1]See Page 256 in [6]

residual spectrum, we have $\operatorname{Ker}(T^* - \lambda) = \operatorname{Ker}(T - \lambda) = \{0\}$. Thus $\operatorname{Ran}(T - \lambda)$ is dense by Theorem 1.2.11. Thus by definition of the residual spectrum, $\lambda \notin \sigma_r(T)$.

Let f, g be eigenvectors of T such that $Tf = \lambda f$, $Tg = \mu g$ with $\lambda \neq \mu$ being real numbers. We have by self-adjointness

$$(\lambda - \mu)\langle f, g\rangle = \langle Tf, g\rangle - \langle f, Tg\rangle = \langle f, Tg\rangle - \langle f, Tg\rangle = 0.$$

The proof is complete. ■

The following theorem is a version of Weyl's criterion on the spectrum of self-adjoints operators: any $\lambda \in \sigma(T)$ is approximately an eigenvalue of T, that is for any $\varepsilon > 0$, there is $u \in D(T)$ such that $\|(T - \lambda)u\| < \varepsilon\|u\|$.

Theorem 2.1.2 *Let T be a self-adjoint operator. Then a real number λ belongs to $\sigma(T)$ if and only if there exists a sequence $(x_n)_n \subset D(T)$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(T - \lambda)x_n\| = 0$.*

Proof

For the " $\Rightarrow$ " part, since $\sigma_r(T) = \emptyset$, there are two cases: either $\operatorname{Ker}(T - \lambda) \neq \{0\}$ or $\operatorname{Ker}(T - \lambda) = \{0\}$ and $\operatorname{Ran}(T - \lambda)$ is a dense proper subspace of $\mathcal{H}$. In the first case, we take x_n for all n with $\|x_n\| = 1$ and $Tx_n = \lambda x_n$, which are normalized eigenvectors associated to λ, with n being either finite or infinite depending on the geometric multiplicity of λ. In the second case, $\operatorname{Ran}(T - \lambda)$ is a dense proper subspace of $\mathcal{H}$. Hence $(T-\lambda)^{-1}$ with domain $\operatorname{Ran}(T-\lambda)$ is an unbounded operator[2]. In particular, there is $g_n \in \operatorname{Ran}(T - \lambda)$ with $\|g_n\| = 1$ but $\|(T - \lambda)^{-1}g_n\| \to \infty$ as $n \to \infty$. We may take

$$x_n = \big((T - \lambda)^{-1}g_n\big)\|(T - \lambda)^{-1}g_n\|^{-1}.$$

For the " $\Leftarrow$ " part, assume $\lambda \in \rho(T)$. Then there is C such that for all $u \in \mathcal{H}$

$$\|R_T(\lambda)u\| \leq C\|u\|.$$

Let $\{x_n\}$ be such a sequence. Then

$$1 = \|x_n\| = \|R_T(\lambda) \circ (T - \lambda)x_n\| \leq C\|(T - \lambda)x_n\| \to 0,\ n \to \infty.$$

This contradiction implies $\lambda \in \sigma(T)$. The proof is complete. ■

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[2]Recall a bounded operator has its domain equal to the whole space.

2.2 Fundamental criteria for self-adjointness

Theorem 2.2.1 (Basic Criteria for self-adjointness) *Let T be a symmetric operator. Then the following three statements are equivalent:*

(1). *T is self-adjoint;*

(2). *T is closed and* $\mathrm{Ker}(T^* \pm i) = \{0\}$*;*

(3). $\mathrm{Ran}(T \pm i) = \mathcal{H}$.

Proof

(1)⇒(2) is clear. We show (2) ⇒ (3). Since $\mathrm{Ker}(T^* + i) = \{0\}$, we have $\mathrm{Ran}(T - i)$ is dense. Moreover, it is closed since T is closed. Therefore, $\mathrm{Ran}(T - i) = \mathcal{H}$. Similarly, $\mathrm{Ran}(T + i) = \mathcal{H}$.

Next, we show (3) ⇒ (1). We must show $D(T) = D(T^*)$, since T is symmetric. First of all, we have $D(T) \subset D(T^*)$ by definition. Let $g \in D(T^*)$. By using (3), we may find $f \in D(T)$ such that $(T - i)f = (T^* - i)g$. Notice that T^* is an extension of T, we have $(T^* - i)(f - g) = 0$, meaning that $f - g \in \mathrm{Ker}(T^* - i) = (\mathrm{Ran}(T + i))^\perp = \{0\}$. Thus $D(T^*) = D(T)$. The proof is complete. ■

Remark 2.2.2 *If T is a closed symmetric operator, we define its* deficiency subspaces *by*

$$\mathcal{L}^\pm := \{f \in D(T^*) \mid T^* f = \pm i f\}.$$

The deficiency indices of T *, $\mathfrak{D}^\pm(T)$, are the dimensions of the deficiency subspaces. Then the closed symmetric operator T is self-adjoint if and only if $\mathfrak{D}^\pm(T) = 0$.*

Example 5 *The Laplacian Δ on $H^2(\mathbb{R}^n)$ is self-adjoint. In fact, Δ is symmetric and closed on $H^2(\mathbb{R}^2)$, but it is not closed on $C_0^\infty(\mathbb{R}^n)$. The closure of $(\Delta, C_0^\infty(\mathbb{R}^n))$ is $(\Delta, H^2(\mathbb{R}^n))$. The spectrum of the self-adjoint operator $-\Delta$ on $H^2(\mathbb{R}^2)$ is $\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0, \infty)$.*

Quite often in practice, the notion of self-adjointness is somehow restrictive and it is necessary to generalize it in certain way for applications. In the above example, the Laplacian Δ on $C_0^\infty(\mathbb{R}^n)$ is not self-adjoint but its closure. This leads to the following conception.

Recall that we say an operator T is *essentially self-adjoint* if it is symmetric and its closure is self-adjoint. Any domain with this property is called a *core* for the corresponding self-adjoint operator.

Corollary 2.2.3 *Let T be a symmetric operator on a Hilbert space. Then the following are equivalent.*

1) T is essentially self-adjoint

2) $\mathrm{Ker}(T^* \pm i) = \{0\}$.

3) $\mathrm{Ran}(T \pm i)$ *are dense.*

Every non-negative symmetric operator T has at least one non-negative self-adjoint extension. If T is not essentially self-adjoint, then this extension is called the *Friedrichs* extension.

Theorem 2.2.4 (Friedrichs-Lax-Milgram) *Let Q be a quadratic form defined on the domain $D(T)$ of a non-negative symmetric operator T defined by $Q(f,g) = \langle Tf, g\rangle$. Then Q is closable and its closure is associated with a self-adjoint extension of T.*

The proof of this theorem is beyond the scope of the course. We refer to [2]

Let us end up this section with the notion of the Dirichlet Laplacian. Let Ω be a region in $\mathbb{R}^n$. The initial domain of $-\Delta$ on Ω is defined as $C_0^\infty(\overline{\Omega})$ which consists of smooth functions on Ω all of whose partial derivatives can be extended continously to the closure of Ω, moreover, which vanish on the boundary of Ω.

Under the above conditions, the operator $-\Delta$ is symmetric and non-negative, with the associated quadratic form being closable on the domain $C_0^\infty(\overline{\Omega})$

$$Q(f,g) := \int_\Omega \sum_{j=1}^n \frac{\partial f(x)}{\partial x_j} \overline{\frac{\partial g(x)}{\partial x_j}} dx,$$

in view of the Friedrichs-Lax-Milgram extension theorem. We shall use $-\Delta_D$ to refer to the self-adjoint operator associated with the closure of the quadratic form Q. We call $-\Delta_D$ the Dirichlet Laplacian.

The domain of Q reads $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ for the H^1 norm. Furthermore, $-\Delta_D$ is the unique non-negative self-adjoint operator on $L^2(\Omega)$ such that $\langle (-\Delta_D)^{1/2} f, (-\Delta_D)^{1/2} g\rangle = Q(f,g)$ for all $f, g \in H_0^1(\Omega)$.

Chapter 3

The essential spectrum and Compact operators

3.1 Weyl's criterion on the essential spectrum

A sequence $\{x_n\}$ in a Hilbert space is said to converge *weakly* to $x \in \mathcal{H}$ if for each $v \in \mathcal{H}$, we have $\langle x_n, v\rangle \to \langle x, v\rangle$ as $n \to \infty$, and we write $x_n \underset{n\to\infty}{\rightharpoonup} x$.

Recall in Theorem 2.1.2, for a self-adjoint operator T, a real number λ belongs to $\sigma(T)$ or not is characterized by a sequence $\{x_n\}$ in $D(T)$ such that $\|x_n\| = 1$ and $\|(T - \lambda)x_n\| \to 0$. If $\lambda \in \sigma_d(T)$, then it is an eigenvalue of T with finite multiplicity. Therefore, such a sequence $\{x_n\}$ associated to λ can not contains any subsequence converging weakly to zero. If for a given λ, to which there associated such a sequence which converges weakly to zero[1], then one may suspect that λ is contained in the essential spectrum of T. This is the idea of Weyl's criterion on $\sigma_{ess}(T)$ for a self-adjoint operator.

Definition 3.1.1 *A sequence $(x_n)_n \subset \mathcal{H}$ is called a* Weyl sequence *for T and z if $x_n \in D(T)$ such that $\|x_n\| = 1$, $x_n \underset{n\to\infty}{\rightharpoonup} 0$ and $\lim_{n\to\infty} \|(T - z)x_n\| = 0$.*

Theorem 3.1.2 (Weyl's criterion) *Let T be a self-adjoint operator on $\mathcal{H}$.Then $z \in \sigma_{ess}(T)$ if and only if there exists a Weyl sequence for T and z.*

Proof

Let $z \in \sigma_{ess}(T)$. There are two cases, either $\mathrm{Ker}(T - z) \neq \{0\}$ or $\mathrm{Ker}(T - z) = \{0\}$ and $\mathrm{Ran}(T - z)$ is a dense proper subspace of $\mathcal{H}$. In the first case, the multiplicity[2]

[1] In other words, the essential spectrum is really a character linked to the infinite dimensional space, where he sequence associated to the essential spectrumcan be made orthogonal. .

[2] For self-adjoint operators, the geometric and algebraic multiplicity are the same.

of z is infinite and there is a sequence of eigenvectors $\|x_n\| = 1$ containing a subsequence converging weakly to zero. In the second case, $T - z$ has an unbounded inverse defined on a dense domain and hence there is $g_n \in \operatorname{Ran}(T-z)$ with $\|g_n\| = 1$ but $\|(T-z)^{-1}g_n\| \to \infty$ as $n \to \infty$. We may take

$$x_n = \big((T-z)^{-1}g_n\big)\|(T-z)^{-1}g_n\|^{-1}.$$

Then clearly $\|x_n\| = 1$ and $\|(T-z)x_n\| \to 0$. It remains to show that $x_n \xrightarrow[n\to\infty]{} 0$. For any $f \in \mathcal{H} \cap D(((T-z)^{-1})^*)$, we have by the Schwarz inequalty

$$|\langle x_n, f\rangle| = \|(T-z)^{-1}g_n\|^{-1}|\langle g_n, ((T-z)^{-1})^* f\rangle| \le \|(T-z)^{-1}g_n\|^{-1}\|((T-z)^{-1})^* f\|$$

where the right hand side goes to zero as $n \to \infty$. By using the basic criteria for self-adjointness, one may prove that $D(((T-z)^{-1})^*)$ is dense. This completes the proof of the " $\Rightarrow$ " part.

For the " $\Leftarrow$ " part. Let $\{x_n\}$ be a Weyl sequence for (T, z) By Theorem 2.1.2, $z \in \sigma(T)$. It remains to show $z \notin \sigma_d(T)$. This part of proof is much more involved. We first claim the following fact:

$$(*) \quad \boxed{z \in \sigma_d(T) \Leftrightarrow \dim \operatorname{Ker}(T-z) < \infty, \ \ (T-z)|_{(\operatorname{Ker}(T-z))^\perp \cap D(T)} \text{ is invertible.}}$$

Assume this claim $(*)$. Consider $\dim \operatorname{Ker}(T-z) = \infty$ and $\dim \operatorname{Ker}(T-z) < \infty$. In the first case, $z \in \sigma_{ess}(T)$ and nothing to prove. In the second case, we need to show that $T_z := (T-z)|_{(\operatorname{Ker}(T-z))^\perp \cap D(T)}$ is not invertible, that is T_z admits no bounded inverse.

Let $\{e_k\}_{k=1}^M$ be the orthonormal basis for $\operatorname{Ker}(T-z)$ and let $\mathbb{P}$ be the orthogonal projection onto $\operatorname{Ker}(T-z)$. Then by weak convergence

$$\|\mathbb{P}x_n\|^2 = \sum_{k=1}^M |\langle x_n, e_k\rangle|^2 \to \infty, n \to \infty.$$

Therefore, $\|(1-\mathbb{P})x_n\| \to 1$. Let $u_n = (1-\mathbb{P})x_n\|(1-\mathbb{P})x_n\|^{-1}$. Then $\|u_n\| = 1$ and

$$\|T_z u_n\| = \|(1-\mathbb{P})x_n\|^{-1}\|(T-z)x_n\| \to 0,$$

since $\{x_n\}$ is a Weyl sequence. To complete the proof, it only needs to observe now T_z^{-1} is unbounded along the sequence u_n. ■

For the claim $(*)$, we do not have enough time to provide a complere proof of this theorem since it involves additional implements, especially the Riesz projection. We refer to Theorem of [5] for a proof.

Example 6 *The essential spectrum of* $-\Delta_D$ *on* $L^2_{rad}(\Omega)$ *where* $\Omega = \mathbb{R}^3 \backslash B(0,1)$ *and* $L^2_{rad}(\Omega)$ *is the closure of radial* $C_0^\infty(\Omega)$ *functions in* L^2-*norm. Since* $\langle -\Delta_D f, f\rangle \geq 0$ *for all* $f \in D(-\Delta_D)$. *We have by Theorem 2.1.2,* $\sigma(-\Delta_D) \subset [0,\infty)$. *We may use Weyl's criterion to show that if* $\lambda > 0$, *then* $\lambda \in \sigma_{ess}(-\Delta_D)$, *which yields* $[0,\infty) \subset \sigma(-\Delta_D)$ *since the essential spectrum is a closed set. This implies* $\sigma_{ess}(-\Delta_D) = [0,\infty)$. *We need to construct a Weyl sequence for* $(-\Delta_D, \lambda)$. *Let* $0 < \varepsilon < 2^{-10}\lambda^{1/2}$ *and consider for* $r > 1$

$$f_{\lambda^{1/2},\varepsilon}(r) := \frac{\cos\big((\lambda^{1/2}-\varepsilon)(r-1)\big) - \cos\big((\lambda^{1/2}+\varepsilon)(r-1)\big)}{(\pi\varepsilon)^{\frac{1}{2}} r(r-1)}.$$

Then we have $f_{\lambda^{1/2},\varepsilon}(1) = 0$, $\|f_{\lambda^{1/2},\varepsilon}\| = 1$ *and* $f_{\lambda^{1/2},\varepsilon}$ *converges weakly to zero. Moreover,* $\|(-\Delta - \lambda) f_{\lambda^{1/2},\varepsilon}\| \to 0$ *as* $\varepsilon \to 0$. *Thus, taking a sequence of* $\varepsilon_n \to 0$ *as* $n \to \infty$, *we obtain a Weyl sequence for* $(-\Delta_D, \lambda)$.

Exercise 7 *We shall use the Dirichlet integral reads* $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Let $0 < \varepsilon < 2^{-10}\lambda_0$ *and consider for* $r > 1$

$$f_{\lambda_0,\varepsilon}(r) := \frac{\cos\big((\lambda_0-\varepsilon)(r-1)\big) - \cos\big((\lambda_0+\varepsilon)(r-1)\big)}{(\pi\varepsilon)^{\frac{1}{2}} r(r-1)}.$$

Compute

$$\|f_{\lambda_0,\varepsilon}\|^2 := \int_1^\infty |f_{\lambda_0,\varepsilon}(r)|^2 r^2 dr = \frac{I(\varepsilon)}{\pi\varepsilon}$$

where

$$I(\varepsilon) = \int_0^\infty \frac{|\cos\big((\lambda_0+\varepsilon)\rho\big) - \cos\big((\lambda_0-\varepsilon)\rho\big)|^2}{\rho^2} d\rho.$$

First, notice that $I(0) = 0$ *and*

$$\frac{d}{d\varepsilon} I(\varepsilon) = 2\int_0^\infty \Big(\cos\big((\lambda_0-\varepsilon)\rho\big) - \cos\big((\lambda_0+\varepsilon)\rho\big)\Big)\Big(\sin\big((\lambda_0-\varepsilon)\rho\big) + \sin\big((\lambda_0+\varepsilon)\rho\big)\Big)\frac{d\rho}{\rho}.$$

Now, use

$$\sin\alpha + \sin\beta = 2\sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big)$$
$$\cos\alpha - \cos\beta = 2\sin\Big(\frac{\alpha+\beta}{2}\Big)\sin\Big(\frac{\beta-\alpha}{2}\Big)$$

to simplify

$$\begin{aligned}\frac{d}{d\varepsilon} I(\varepsilon) &= 2\int_0^\infty \Big(1 - \cos(2\lambda_0\rho)\Big)\frac{\sin(2\varepsilon\rho)}{\rho} d\rho \\ &= 2\int_0^\infty \frac{\sin x}{x} dx - \int_0^\infty \frac{\sin(2(\lambda_0+\varepsilon)\rho) - \sin(2(\lambda_0-\varepsilon)\rho)}{\rho} d\rho \\ &= \pi.\end{aligned}$$

We have $\|f_{\lambda_0,\varepsilon}\| = 1$.

By the fundamental theorem of calculus, we have

$$f_{\lambda_0,\varepsilon}(r) = r^{-1}\sqrt{\frac{\varepsilon}{\pi}}\int_{-1}^{1}\sin\Big((\lambda_0+\theta\varepsilon)(r-1)\Big)d\theta \to 0, \varepsilon \to 0.$$

Thus for all radial $g \in C_c^\infty(\Omega)$, *we have* $\langle f_{\lambda_0,\varepsilon}, g\rangle \to 0$ *as* $\varepsilon \to 0$ *by using the dominated convergence theorem. This proves the weak convergence.*

Finally, prove that $\|(-\Delta - \lambda_0^2)f_{\lambda_0,\varepsilon}\| \to 0$ *as* $\varepsilon \to 0$. *Observe*

$$(-\Delta - \lambda_0^2)f_{\lambda_0,\varepsilon}(r) = \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon}(\lambda^2-\lambda_0^2)\frac{\sin\big(\lambda(r-1)\big)}{(\pi\varepsilon)^{\frac{1}{2}}r}d\lambda$$

which is continuous and $\sim O(\lambda_0^2\sqrt{\varepsilon})$ *when* $r \to 1$. *For* r *large, change variables* $\lambda \to \varepsilon\lambda + \lambda_0$ *and integration by parts to get an upper bound* $O(\sqrt{\varepsilon}r^{-2})$.

3.2 Compact operators

Definition 3.2.1 *A bounded linear operator* K *on* $\mathcal{H}$ *is called compact if it maps any weakly convergent sequence into a strongly convergent sequence. The set of compact operators on* $\mathcal{H}$ *is denoted by* $\mathcal{K}(\mathcal{H})$.

Example 7 *Let* $K(x,y)$ *be a continuous function on* $[0,1]\times[0,1]$. *Then an operator defined as*

$$C([0,1]) \ni f(x) \mapsto (Kf)(x) = \int_0^1 K(x,y)f(y)dy$$

extended to $L^2([0,1])$, *is a compact operator.*

Theorem 3.2.2 (Riesz-Schauder) *Let* $\mathcal{H}$ *be a Hilbert space and* $T \in \mathcal{K}(\mathcal{H})$. *Then the spectrum* $\sigma(T)$ *of the compact operator* T *consists of non-zero isolated eigenvalues of finite multiplicity. If* $z_n \in \sigma(T)$ *is a sequence of distinct eigenvalues such that* $\lim_{n\to\infty} z_n = z$, *then* $z = 0$. *Furthermore, if* $\dim \mathcal{H} = \infty$, *then* $0 \in \sigma(T)$.

We refer to [1] for a proof of this theorem. See also Theorem VI.15 in [6].

Lemma 3.2.3 *Let* T *be self-adjoint. Then we have*

(1). $(\mathrm{Ker}\ T)^\perp$ *is an* $T-$*invariant subspace, namely*

$$T : (\mathrm{Ker}\ T)^\perp \cap D(T) \to (\mathrm{Ker}\ T)^\perp$$

(2). The restriction of T to $(\mathrm{Ker}\, T)^{\perp}$ *is self-adjoint and has trivial kernel* $\{0\}$.

Exercise 8 *Prove Lemma 3.2.3*

Theorem 3.2.4 (Closed range) *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{K}(\mathcal{H})$. If $-1 \in \sigma(T)$, then* $\mathrm{Ran}\,(T+1)$ *is closed and* $\mathrm{Ran}\,(T+1) = \big(\mathrm{Ker}\,(T^* + 1)\big)^{\perp}$.

Proof

Let $A = T + 1$ and set

$$G = \mathfrak{G}(A), \quad L = \mathcal{H} \times \{0\}.$$

Then clearly we have

$$G + L = \mathcal{H} \times \mathrm{Ran}\, A; \quad G^{\perp} \cap L^{\perp} = \{0\} \times \mathrm{Ker}\,(A^*),$$

and L is a closed subspace in $\mathcal{H} \times \mathcal{H}$. We need to show $G + L = (G^{\perp} \cap L^{\perp})^{\perp}$.

Step 1. We prove G is a closed subspace.

We only need to show that $\mathrm{Ran}\, A$ is closed. By Riesz-Schauder's theorem, we have $\dim \mathrm{Ker}(T+1) < \infty$. Then

$$\mathcal{H} = \mathrm{Ker}\, A \oplus M,$$

where M is a closed subspace of $\mathcal{H}$. Indeed, let $\{e_j\}_{j=1}^{k}$ be a bisis for $\mathrm{Ker}\, A$ so that

$$\mathrm{Ker}\, A = \{x \in \mathcal{H} \mid x = a_1(x)e_1 + \cdots + a_k(x)e_k\}.$$

For each $j \in \{1, 2, \ldots, k\}$, a_j extends to be a linear functional on $\mathcal{H}$ (the Hahn-Banach theorem). We may take $M = \cap_{j=1}^{k} \mathrm{Ker}\, a_j$ where each $\mathrm{Ker}\, a_j$ is a closed subspace by continuity of a_j.

Let $B = A|_M$. Then $\mathrm{Ker}\, B = \{0\}$ and $\mathrm{Ran}\, B = \mathrm{Ran}\, A$. To show that $\mathrm{Ran}\, B$ is closed, it suffices to show that there exists a $\gamma > 0$ such that

$$\gamma \|x\| \leq \|Bx\|, \qquad \forall\, x \in M. \tag{3.1}$$

Clearly (3.1) implies $\{x_n\}$ is a Cauchy sequence provided that $\{Bx_n\}$ is.

To show (3.1), we draw by contradiction: assume for each $\gamma > 0$, there is a sequence $\{x_n\} \in M$ with $\|x_n\| = 1$ and $\lim\limits_{n\to\infty} \|Bx_n\| = 0$. By compactness of T, there is a subsequence of $\{x_n\}$, still denoted as $\{x_n\}$ such that Tx_n converges to some x_0. We have $x_n \to -x_0$ as $n \to \infty$. Thus $x_0 \in M$ by closedness, and we have

$$Bx_0 = -\lim_{n\to\infty} Bx_n = 0.$$

This implies $x_0 = 0$ On the other hand, by the strong convergence, we have $\|x_0\| = 1$, which is a contradiction.

Step 2. Notice that $((G+L)^\perp)^\perp = G+L$ since G and L are closed. To show $G+L = (G^\perp \cap L^\perp)^\perp$ is equivalent to

$$(G+L)^\perp = G^\perp \cap L^\perp, \tag{3.2}$$

which follows immediately from the definition. The proof is complete. ■

The following theorem is one of the various versions of the so-called Fredholm alternative. It is formulated in a fashion convenient particularly for later use in the next chapter.

Theorem 3.2.5 (Fredholm Alternative) *Let K be a compact operator. Then we have*

(1). The equation $f + Kf = g$ has a unique solution for every $g \in \mathcal{H}$ if and only if $-1 \notin \sigma(K)$.

(2). Assume $-1 \in \sigma(K)$. Then the equation $f + Kf = g$ has a unique solution if and only if $g \in \big(\mathrm{Ker}(1+K^*)\big)^\perp$.

Proof

We first show (1) . "$\Rightarrow$" : If $-1 \in \sigma(K)$, then $f + Kf = 0$ has a solution in $\mathcal{H} \setminus \{0\}$. By Riesz-Schauder theorem, -1 is an eigenvalue of K. Therefore $\mathrm{Ker}(K+1) \neq \{0\}$ and there exists an $f \in \mathcal{H} \setminus \{0\}$ such that $f + Kf = 0$. Next, for "$\Leftarrow$": since $-1 \in \rho(K) = \mathbb{C} \setminus \sigma(K)$, we may set $f = R_K(-1)g$ for every $g \in \mathcal{H}$.

We now prove (2). "$\Rightarrow$" : for any $h \in \mathrm{Ker}(1+K^*)$, we have

$$(g,h) = (f + Kf, h) = (f, h + K^*h) = 0.$$

Thus we have $g \in \big(\mathrm{Ker}(1+K^*)\big)^\perp$. Next, to show "$\Leftarrow$", we let $K_1 = K\big|_{(\mathrm{Ker}(K+1))^\perp}$. We claim that $-1 \in \rho(K_1)$. Assume $-1 \in \sigma(K_1)$, then -1 is an eigenvalue of K_1 as well by the Riesz-Schauder theorem. There is $f \in (\mathrm{Ker}(K+1))^\perp \setminus \{0\}$ such that $K_1 f + f = 0$, which implies $f \in \mathrm{Ker}(K+1)$. However, since $(\mathrm{Ker}(K+1))^\perp \cap \mathrm{Ker}(K+1) = 0$, we must have $f = 0$, which is a contradiction. Thus $1 + K$ restricted to $(\mathrm{Ker}(K+1))^\perp$, denoted as A, has a bounded inverse $R_{K_1}(-1)$. Moreover, by the closed range theorem, we have

$$\mathrm{Ran}\, A = (\mathrm{Ker}(1+K^*))^\perp.$$

We may take $f = R_{K_1}(-1)g$ for every $g \in (\mathrm{Ker}(1 + K^*))^\perp$ as the unique solution. The proof is complete. ■

See Theorem VI.14 in [6] for an analytic Fredholm theorem, and Theorem XIII.13 in [7] for a meromorphic Fredholm theorem.

Chapter 4

Stability of the essential spectrum under relatively compact perturbations

4.1 Relatively compact operators

Definition 4.1.1 *Let T be a closed operator with $\rho(T) \neq \emptyset$. An operator S is called* relatively $T-$compact *if $D(T) \subset D(S)$ and $S \circ R_T(z)$ is compact for some $z \in \rho(T)$*

Exercise 9 *Show that in the above definition that given the assumption of the compactness of $S \circ R_T(z)$ for one $z \in \rho(T)$ implies that $S \circ R_T(z)$ is compact for all $z \in \rho(T)$*

[Hint: use the first resolvent identity]

4.2 Stability of the Essential Spectrum: Weyl's Theorem

Lemma 4.2.1 *Let T and S be self-adjoint operators, and assume that $S - T$ is $T-$compact. Then $S - T$ is S-compact.*

Proof

The proof is divided into three steps.

Step 1. Denote by $V = S - T$. We show $-1 \notin \sigma(V \circ (T - i)^{-1})$ by contradiction argument.

Let $K = V \circ (T - i)^{-1}$ and assume that $-1 \in \sigma(K)$. Since K is compact and $0 \in [\mathrm{Ker}(1 + K^*)]^{\perp}$, we have from Fredholm alternative Theorem 3.2.5, there is a

$\phi \in \mathcal{H} \setminus \{0\}$ [1] such that $\phi + K\phi = 0$. This is true if and only if

$$S \circ (T - i)^{-1}\phi = i(T - i)^{-1}\phi.$$

However, S is self-adjoint, we have by Theorem 2.1.1,

$$(T - i)^{-1}\phi = 0,$$

which implies $\phi = 0$.

Step 2. We show

$$(S - i)^{-1} = (T - i)^{-1} \circ [(S - T) \circ (T - i)^{-1} + 1]^{-1}. \tag{4.1}$$

From the second resolvent inequality

$$(T - i)^{-1} - (S - i)^{-1} = (S - i)^{-1} \circ (S - T) \circ (T - i)^{-1},$$

we have

$$(T - i)^{-1} = (S - i)^{-1} \circ [(S - T) \circ (T - i)^{-1} + 1].$$

By Step 1, we see $(S - T) \circ (T - i)^{-1} + 1$ is invertible and (4.1) follows.

Step 3. End of the proof. First, we note that $D(S) = D(S - T) \cap D(T) \subset D(S - T)$. Next, from (4.1), we have

$$(S - T) \circ (S - i)^{-1} = (S - T) \circ (T - i)^{-1} \circ [(S - T) \circ (T - i)^{-1} + 1]^{-1}.$$

Since the right side of the above equation is by assumption a compact operator, the proof is complete. ■

Theorem 4.2.2 (Weyl) *Let T and S be self-adjoint operators, and assume that $S - T$ is $T-$compact. Then we have*

$$\sigma_{ess}(T) = \sigma_{ess}(S).$$

Proof

First of all, we note that by Lemma 4.2.1, $S - T$ is also $S-$compact. Thus, it suffices to show $\sigma_{ess}(T) \subset \sigma_{ess}(S)$ by symmetry of the role of T and S in the argument. To this end, in view of Weyl's criterion Theorem 3.1.2, it suffices to show that for each $\lambda \in \sigma_{ess}(T)$, the Weyl sequence $\{w_n\}$ for (T, λ) is also a Weyl sequence for (S, λ).

[1] From (2) of Theorem 3.2.5, we obtain the existence of $\phi \in \mathcal{H}$, and from (1) of Theorem 3.2.5, we may take $\phi \neq 0$.

We need to show $w_n \in D(S)$ and $\lim_{n\to\infty} \|(S-\lambda)w_n\| = 0$. From the assumption that $S-T$ is $T-$compact, we have $D(T) \subset D(S-T)$. Thus, we have $D(S) = D(T) \cap D(S-T) = D(T)$ and we have $\{w_n\} \subset D(S)$.

Next, we show the strong convergence to zero of the sequence $\{(S-\lambda)w_n\}$. Write

$$(S-\lambda)w_n = (S-T)\circ(T-i)^{-1}\circ(T-i)w_n + (T-\lambda)w_n$$

where $\lim_{n\to\infty} \|(T-\lambda)w_n\| = 0$ since $\{w_n\}$ is a Weyl sequence for (T,λ). Notice that

$$(T-i)w_n = (T-\lambda)w_n + (\lambda - i)w_n \xrightharpoonup[n\to\infty]{} 0,$$

we have by the compactness of $(S-T)\circ(T-i)^{-1}$,

$$\lim_{n\to\infty} \|(S-T)\circ(T-i)^{-1}\circ(T-i)w_n\| = 0.$$

The proof is complete. ■

For various forms of generalisations of this theorem, please see [7].

4.3 An application

In this section, we apply the theory of this course to deduce an auxiliary result used in a work [3].

Let $\Omega = \mathbb{R}^3 \setminus B(0,1)$ and V be a real-valued, smooth radial function on Ω such that $V(1) = 0$ and behaves like $|x|^{-\kappa}$ for $r = |x|$ large with $\kappa > 1$. We denote by

$$L_V = -\Delta - V(x)\ , \tag{4.2}$$

We study the essential spectrum of L_V.

Denote by

$$D(-\Delta)_{rad} = \{f \in D(-\Delta) : f(x) \text{ radial}\}$$

as the domain of the Dirichlet Laplacian $-\Delta_D$ on $L^2_{rad}(\Omega)$. We have proved in Example 6 that the spectrum of $-\Delta_D$ is $[0,+\infty)$. We are about to show that L_V and $-\Delta_D$ have the same essential spectrum. Notice that L_V is self-adjoint with the same domain of $-\Delta_D$. By Weyl's theorem (Theorem 4.2.2), it suffices to show that V is relatively $-\Delta$-compact.

Let $M_V : f \mapsto Vf$ denote the multiplication operator by V. We need to show $M \circ (-\Delta_D + i)^{-1}$ is a compact operator.

Let $\{f_n\} \subset L^2_{rad}(\Omega)$ be a sequence of functions which converges weakly to zero. By the uniform boundedness principle, we have in particular that f_n is bounded and we may assume $\|f_n\|_2 \leq 1$.

Let $g_n = (-\Delta_D + i)^{-1} f_n$. Then by the boundedness of the resolvent, we have $\|g_n\|_2 \leq C$ for some constant $C > 0$. Moreover, by multiplying both sides of $(-\Delta + i)g_n = f_n$ by $\overline{g_n}$ and integration by parts, we have

$$\|\nabla g_n\|_2^2 \leq \left| \mathbf{Re} \int_\Omega f(x)\overline{g_n(x)}dx \right| \leq \|f_n\|_2 \times \|g_n\|_2 \leq C,$$

where we used the Cauchy-Schwarz inequality.

It suffices to prove that $\{Vg_n\}$ converges strongly in L^2. Equivalently, we need to show $\{Vg_n\}$ is a Cauchy sequence. We need a radial Sobolev embedding

$$\sup_{r>1} r^{1/2}|g(r)| \leq \|\nabla g\|_2. \tag{4.3}$$

For any $\varepsilon > 0$, we take R large enough such that by (4.3)

$$\int_R^\infty |V(r)g_n(r)|^2 r^2 dr \leq C \int_R^\infty |V(r)|^2 r dr \leq C' \int_R^\infty r^{-2\kappa+1} \leq C'' R^{-2(\kappa-1)} < \varepsilon^2/9$$

holds for all n. On the ball $B(0,R)$, there exists $C_0 > 0$ such that we have $\sup_{1<r<R} |V(r)| \leq C_0$ by smoothness of V. By using Rellich's compact embedding theorem[2]: the identity operator $\mathrm{id} : H^1(K) \to L^2(K)$ is compact if $K \subset \mathbb{R}^3$ is an open bounded Lipschitz domain. Thus there is N such that for all $n, n' \geq N$ we have $\|g_n - g_{n'}\|_{L^2(B(0,R))} < \frac{\varepsilon}{3C_0}$. Therefore, we have for all $n, n' \geq N$

$$\|Vg_n - Vg_{n'}\|_2 \leq C_0 \|g_n - g_{n'}\|_{L^2(B(0,R))} + \|Vg_n\|_{L^2(r \geq R)} + \|Vg_{n'}\|_{L^2(r \geq R)} < \varepsilon.$$

The proof is complete.

Exercise 10 *Prove the radial Sobolev embedding* (4.3).

[2]We refer to [1] for a proof

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